Lecture 4.

Sequences and completeness.

Let (X, d) be a metric space.

Def. ① A sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to $x_0 \in X$, $\lim_{n\to\infty} x_n = x$ or $x_n \to x_0$, if $\forall \varepsilon > 0\ \exists N$ s.t. $n \geq N \Rightarrow d(x_n, x_0) < \varepsilon$ ($\Leftrightarrow d(x_n, x_0) \to 0$ in $\mathbb{R}$).

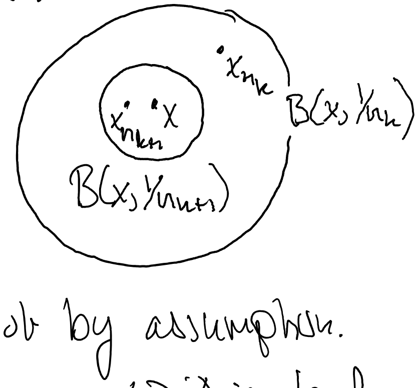
② If $A \subseteq X$, then x_0 is a limit point of A if $\exists$ sequence $\{x_n\}_{n=1}^{\infty}$ of distinct points in A s.t. $x_n \to x_0$.

Prop 1 (i) $A \subseteq X$ is closed $\Leftrightarrow$ A contains all its limit points.

(ii) $\overline{A} = A \cup \{$all limit points$\}$.

Pf. (i) $\Rightarrow$: Let $\{x_n\}$ be seq. (not nec. of distinct points) s.t. $x_n \to x_0 \in X$. If $x_0 \notin A$, then (since A is closed $\Leftrightarrow X \setminus A$ open) $\exists B(x_0, \varepsilon) \subseteq X \setminus A$. But then $d(x_n, x_0) \geq \varepsilon$, which cannot be since $x_n \to x_0$. Thus, $x_0 \in A$. (We proved the limit of any seq. in A is contained in A if A is closed.)

$\Leftarrow$: If $x \in X \setminus A$, then x is not a limit point of A. Consider the collection of balls $\{B(x, \frac{1}{n})\}_{n=1}^{\infty}$. If $\nexists N$ s.t. $B(x, 1/n) \cap A = \emptyset$ for $n \geq N$, then we get a sequence $n_1 < n_2 < \dots \to \infty$ and $x_{n_k} \in A \cap B(x, 1/n_k) \setminus B(x, 1/n_{k-1})$.



But then $\{x_{n_k}\}_{k=1}^{\infty}$ is a seq. of distinct points and $x_{n_k} \to x$ as $k \to \infty$, so x would be a limit point of A, which it is not by assumption. Thus, $\exists N$ s.t. $B(x, 1/N) \cap A = \emptyset \Rightarrow X \setminus A$ is open $\Leftrightarrow A$ is closed. ☒

(ii) Exercise.

Def. ① A seq. $\{x_n\}_{n=1}^{\infty}$ in X is a Cauchy sequence if $\forall \varepsilon > 0\ \exists N$ s.t.

$d(x_n, x_m) < \varepsilon \quad \forall n, m \geq N$,

② X is <u>complete</u> if every Cauchy seq. $\{x_n\}_{n=1}^{\infty}$ has a limit in X.

<u>Ex.</u> ① $\mathbb{R}$ is complete (non-trivial but essentially by construction).

② $\mathbb{R} \setminus \{0\}$ is <u>not</u> complete.

③ $\mathbb{C} \cong \mathbb{R}^2$ is complete (indeed, using ① it is easy to see $\mathbb{R}^n$ is complete).

<u>Cantor's Thm.</u> X is complete $\Leftrightarrow$ $\forall \{F_n\}_{n=1}^{\infty}$ non-empty, <u>closed</u> subsets s.t.

$F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$ (nested)

(ii) $\operatorname{diam}(F_n) = \sup_{x,y \in F_n} d(x,y) \longrightarrow 0$ as $n \to \infty$.

it holds that $\exists x_0 \in X$ s.t. $\bigcap_{n=1}^{\infty} F_n = \{x_0\}$.

<u>Pf.</u> $\underline{\Rightarrow}$: Let $\{F_n\}$ be as above. We shall show $\exists x_0$ s.t. $x_0 \in \bigcap_{n=1}^{\infty} F_n$.
Easy to see from (ii) that $\bigcap_{n=1}^{\infty} F_n$ cannot contain more than one point. For each n, we pick $x_n \in F_n \to \{x_n\}_{n=1}^{\infty}$ seq. in X.
$\forall \varepsilon > 0 \; \exists N$ s.t. $\operatorname{diam} F_n < \varepsilon$, $n \geq N$. Thus, if $n, m \geq N$ then by (i) $x_n, x_m \in F_N \Rightarrow d(x_n, x_m) \leq \operatorname{diam} F_N < \varepsilon$. $\Rightarrow$
$\{x_n\}$ is Cauchy. Since X complete $\exists x_0$ s.t. $x_n \to x_0$.
For any N, $x_n \in F_N$ when $n \geq N$. Since F_N is closed, by (Pf of) Prop 1 (i), $x_0 \in F_N$. But N arbitrary $\Rightarrow x_0 \in \bigcap_{n=1}^{\infty} F_n$.

$\underline{\Leftarrow}$: Let $\{x_n\}_{n=1}^{\infty}$ be Cauchy seq. For any n, define
$F_n = \overline{\{x_n, x_{n+1}, \ldots\}}$. Each F_n is closed, nonempty and $\{F_n\}$ satisfies nested prop. (ii). We claim $\operatorname{diam} F_N \to 0$ as $N \to \infty$
first not that $\operatorname{diam}\{x_N, x_{N+1}, \ldots\} = \sup d(x_n, x_m) \to 0$

$n, m \geq N$

since $\{x_n\}$ is Cauchy. Next, in general $\operatorname{diam} \bar{A} = \operatorname{diam} A$ for any subset $A \subseteq X$ (future HW problem). $\Rightarrow$ $\operatorname{diam} \bar{F}_N \to 0$ as $N \to \infty$. By assumption $\exists x_0$ s.t. $x_0 \in \bigcap_{n=1}^{\infty} F_n$. $\Rightarrow$ $d(x_n, x_0) \leq \operatorname{diam} F_n \to 0$ so the Cauchy seq. has a limit; i.e. X is complete.

Prop2. If X is complete, then $A \subseteq X$ is closed $\Leftrightarrow$ (A, d) is complete.

Pf. Easy. ☒